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ABSTRACT

In the research described in this document, the goal was to develop definite theoretical characterizations of understanding in a specific domain of problems. Some performance that provides persuasive evidence of understanding was chosen, and hypotheses about cognitive structures and processes that cause the performance to occur were developed. Performance was analyzed on two tasks in high school geometry, developing proofs and checking proofs. For developing proofs, thinking-aloud protocols were obtained from six students who were interviewed approximately once a week during the year they studied geometry. Proof-checking problems were given to 15 different students during interviews in November and May, and a group of 15 college students also were given instruction on proof checking to ascertain whether it could be acquired. Two characterizations of knowledge that can be adopted as objectives of instruction resulted, one on structural understanding and one in understanding the principle of deductive consequence. (MNS)

LEARNING RESEARCH AND DEVELOPMENT CENTER

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4 Forms of Understanding in Mathematical Problem Solving

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One important contribution of psychological science to education has been to provide concepts and principles for formulating behavioral objectives in much of the school curriculum. As we achieve greater understanding of cognitive processes, the concepts and principles that we develop can be used in formulating objectives of instruction in more cognitive terms. These objectives are in the form of cognitive analyses of instructional tasks; Resnick (this volume) summarizes several examples of such analyses. This chapter presents two examples of cognitive task analysis in some detail.

The theoretical issue studied in this research is the nature of understanding. The goal that students should acquire understanding of concepts and principles of mathematics, rather than merely acquire rote procedures, is widely accepted. However, in comparison to goals involving concrete skills such as computational algorithms, objectives of understanding have not been formulated in definite, specific ways. Instructional objectives of the form, "the students should understand [a concept]" have not been considered sufficiently specific to be assessed behaviorally and therefore have played a decreasing role in instructional design.

In the research described in this chapter, the goal was to develop definite theoretical characterizations of understanding in a specific domain of problems. The general strategy used was to choose some performance that provides persuasive evidence of understanding and to develop hypotheses about cognitive structures and processes that cause the performance to occur. Assuming that the performance does require understanding, then the cognitive structures and processes that bring it about constitute understanding, and hypotheses about those structures and processes are hypotheses about the nature of understanding. If the hypotheses are approximately correct, they describe knowledge that constitutes

understanding, and thus can be adopted as meaningful and definite cognitive objectives for instruction.

The research described here involved analyses of performance on two tasks in the domain of high school geometry. These two analyses illustrate several general points. First, the analyses provide definite hypotheses about understanding, illustrating the feasibility of developing such hypotheses for school tasks and showing the kinds of cognitive objectives for instruction for understanding that can be obtained from such analyses.

A second point is one emphasized by Resnick (this volume). She notes that cognitive scientists have begun to give more attention to processes of acquisition and instructional intervention, following a period of almost exclusive focus on analyses of performance of relatively complex tasks. The examples I discuss here are consistent with that trend, in that some aspects of acquisition and instructional intervention are included in the theoretical analyses, along with questions about characteristics of cognitive structures and processes that constitute understanding.

A third point is that analyses of knowledge structures involved in performing school tasks can address questions of theoretical interest as well as potential instructional utility. The examples I discuss are concerned with two significant theoretical issues in the psychology of understanding.

One example considers knowledge for solving problems with structural understanding. Important discussions by Judd (1908), by Katona (1940), and especially by Wertheimer (1945/1959)[1] distinguished problem solving with understanding from problem solving of a rote, mechanical nature. Previous discussions have consisted mainly of examples that illustrate the phenomenon of structural understanding in compelling ways. A goal of the analysis presented here is to clarify the concept of structural understanding by providing a definite and specific hypothesis about cognitive structures and processes that constitute this kind of understanding in the context of a specific problem domain.

The second example considers understanding of a general formal principle related to solutions of specific problems. This issue has been especially salient in relation to questions of cognitive development, where Piaget's (e.g., 1941/1965) research was interpreted as indicating that young children lack understanding of important general logical and mathematical concepts, but more recent studies (e.g., Gelman & Gallistel, 1978) indicate that significant understanding of those concepts should be attributed to young children. A theoretical problem arises from the obvious fact that this understanding cannot be construed as explicit

[1]Wertheimer's analysis is the most widely known; indeed it would be appropriate to use the label "Wertheimer's problem" to refer to the question of learning to solve problems with understanding. It is a pleasure to note that the XXII International Congress of Psychology, for which the initial version of this chapter was prepared, was held very near in time to the 100th anniversary of Wertheimer's birth, and near his birthplace as well. Wertheimer was born 15 April 1880, in Prague.

knowledge—for example, preschoolers obviously do not know the concept of cardinality in an explicit way. The theoretical problem is how to characterize understanding of a principle that is not explicit. The analysis presented in the second example provides a characterization of knowledge that constitutes one form of implicit understanding of a formal principle.

A fourth general point illustrated in these examples is that understanding is not a uniform cognitive state. There are different knowledge structures that constitute understanding in different forms. This is important for theory, because it implies that there will not be a single correct cognitive model of the understanding of a concept or a procedure. It also is important for educational practice, because it implies that goals of teaching for understanding must be formulated in more precise and differentiated terms than is customary in order to play an effective role in instruction. The examples presented, involving structural understanding and implicit understanding of a general principle, involve related but distinct forms of understanding. Both are desirable outcomes of instruction, but adoption of them as instructional objectives would lead to the design of quite different sets of materials and tasks.

The first example includes theoretical analyses of alternative knowledge structures that can be acquired in learning to solve some proof exercises that are included in geometry courses. One alternative has knowledge that constitutes understanding of a structure of relations in the problem: the other lacks that understanding. The alternative that simulates knowledge for understanding provides a definite hypothesis in which problem-solving procedures are integrated with a general schema for part–whole relationships. The analysis also provides a definite hypothesis about cognitive structures that can enable such transfer to occur. In observations reported here, substantial variation was found among students in their performance on a transfer problem that has the same structure as a set of problems that the students had previously learned to solve. In the model that was developed to represent understanding of structure, the schematic knowledge that provides the basis of understanding also provides a basis for transferring problem-solving procedures to novel problems. Thus, the analysis is consistent with the view that ability to transfer knowledge to novel problems provides evidence that students understand the structure that the problems share.

The second example, involving understanding of a formal principle, analyzes implicit understanding of the principle of deductive consequence. This is a metaprinciple in relation to solutions of proof problems; it constitutes a general criterion of validity for proofs. Understanding of the principle involves knowledge about what proofs are, which may be distinct from knowing how to construct correct proofs in specific situations. One form of understanding of deductive consequence is knowledge of the logical requirements for a deductively valid inference. Evidence that a student knows these requirements can be obtained in a task of checking proofs. In this task, the student must distinguish valid proofs from invalid arguments; thus successful performance requires knowledge of the

defining features of valid deduction. In the study reported here, we first found that students taking a high school geometry course did not acquire this understanding to a significant degree. Then an analysis was formulated of a cognitive process that would provide a basis for successful performance on the task of proof checking. This analysis was used in designing instruction that had a beneficial effect on students' performance on proof checking, and thus, I propose, on their understanding of the principle of deductive proof.

STRUCTURAL UNDERSTANDING

The analysis of structural understanding that I conducted was focused on processes of learning and transfer. Discussions of structural understanding such as those by Judd (1908), Katona (1940), and Wertheimer (1945/1959) have distinguished meaningful learning from rote learning of problem-solving procedures. Meaningful learning occurs when the material that is learned is related to some general structure or principle, whereas in rote learning the new procedure or information is simply associated with the specific problem situation in which it is experienced. Evidence for understanding resulting from meaningful learning often is obtained by showing that after meaningful learning, individuals are better able to transfer their knowledge to new kinds of problems.

One of the classical examples of meaningful learning given by Wertheimer is learning the proof in geometry that opposite vertical angles are congruent. Figure 4.1 shows the problem, along with representations of two ways to think about the solution. Wertheimer pointed out that children often learn this proof in a mechanical way, represented by the equations in Fig. 4.1, memorizing the algebraic steps. Wertheimer contrasted this mechanical kind of thinking with a more meaningful version, in which the proof is understood in relation to spatial relationships among the angles. He used a diagram like the one at the bottom of Fig. 4.1 to explain these relationships. The spatial pattern can be seen as a pair of overlapping structures, each involving a pair of angles that form a straight line and containing a shared part.

Empirical Observations

I present some data that illustrate the range of understanding that can occur regarding problems like Fig. 4.1. The data consist of thinking-aloud protocols that were obtained from six students during the year they studied geometry in high school. During the year, each student was interviewed approximately once a week. In each interview, the student solved a few geometry problems and thought aloud as he or she worked on the problems. The major use of this set of data has been in developing a computational model that simulates the problem-

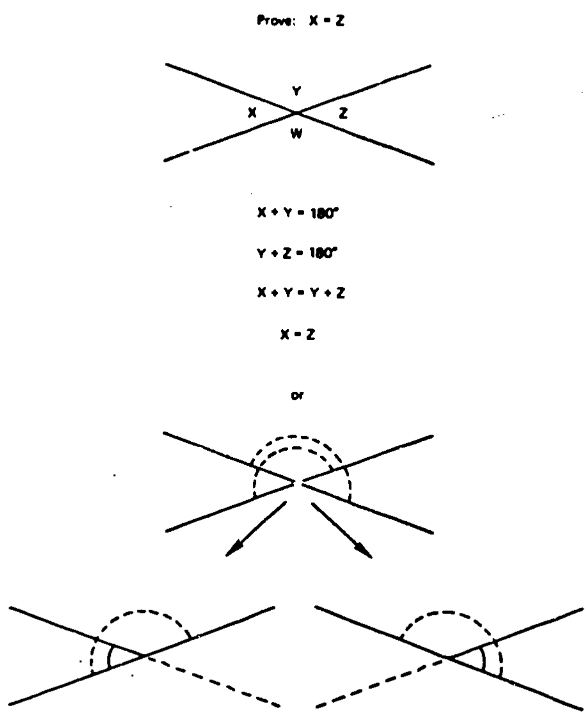


FIG. 4.1. The problem of vertical angles, with two solutions.

solving performance of the students. The model has been described in other reports (Greeno, 1976, 1978; Greeno, Magone, & Chaiklin, 1979).

The third interview, which occurred during the fifth week of the course, included three problems involving the structure of two whole units each composed of two parts, with one of the parts shared by the two wholes. The first problem, given to all six of the students, is shown in Fig. 4.2. A second problem, given to four of the six students after they worked on Fig. 4.2, is about segments, rather than angles. It is shown, along with its solution, in Fig. 4.3. Finally, the problem of vertical angles, shown in Fig. 4.1, was given to four of the six students.

At the time this interview occurred, the students had completed a section of the geometry course in which they learned to solve proof problems involving segments. Thus, problems like Fig. 4.3 were review problems for them; in fact Fig. 4.3 was an example problem in the section that had been completed. They had begun to learn about angles; concepts such as right angles, adjacent angles, and supplementary and complementary angles had been introduced. However, proof problems involving angles had not yet been studied in class. One of the

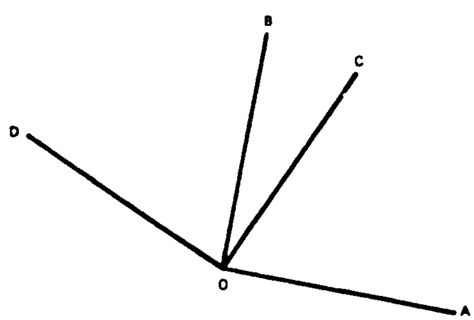
students was working independently and had done some proofs involving angles. For the other five students, problems such as Figs. 4.1 and 4.2 were novel, and presented tests of transfer of knowledge about solving related but distinct problems, such as Fig. 4.3.

To anticipate results that I describe later, the theoretical analysis is a simulation of learning that can occur from example problems involving segments. Two versions of learning were simulated: I refer to these as stimulus–response learning and meaningful learning. The difference between the versions is a hypothesis about the knowledge that enables transfer to occur between problems like Fig. 4.3 and problems like Figs. 4.1 and 4.2.

In both simulations of learning, study of example problems leads to acquisition of knowledge used in solving problems, consisting of procedures for writing lines of proof. The simulations differ in the representations of problems that provide the cues for use of the procedures. In the simulation of stimulus– response learning, problems like Fig. 4.3 are represented quite specifically; the cues for writing lines of proof include segments that are perceived in the diagram of a problem, with features involving the ends of the segments that are needed for the additivity of their lengths.

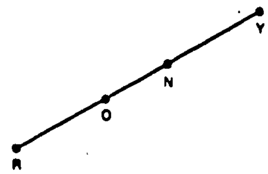
In the simulation of meaningful learning, a more abstract representation is involved. The learner represents the problem using a schema that identifies part–whole relations among the objects in the problem. The problem-solving procedures that are acquired have arguments that are specified in terms of the slots of the part–whole schema. This enables the procedures to be used for problems that have different kinds of objects, providing that the objects in the new problem can be represented using the schema of part–whole relationships.

For example, in stimulus–response learning, a procedure is acquired in which the length of a segment is subtracted from both sides of an equation. This corresponds to Step 6 of the proof in Fig. 4.3. In meaningful learning, the



Given: $\angle AOB$ and $\angle COD$ are right angles
Prove: $\angle AOC = \angle BOD$

FIG. 4.2. A problem used to test transfer.



Given: $\overline{RONY}$
$RN = OY$
Prove: $RO = NY$

Statement	Reason
1. $\overline{RONY}$	1. Given
2. $RN = OY$	2. Given
3. $RN = RO + ON$	3. Segment addition (Step 1)
4. $OY = ON + NY$	4. Segment addition (Step 1)
5. $RO + ON = ON + NY$	5. Substitution (Steps 2, 3, 4)
6. $RO = NY$	6. Subtraction property (Step 5)

FIG. 4.3. A problem from previous learning for students, and the third example problem for simulation of learning.

procedure learned from Step 6 involves subtracting a part that is shared in two whole units from each of them. The conditions for applying the procedure acquired in stimulus–response learning include the existence of segments, whereas the conditions for applying the procedure acquired in meaningful learning include objects designated as parts and wholes. Thus, the procedure acquired in meaningful learning is potentially applicable in a new problem involving angles, whereas the procedure acquired in stimulus–response learning cannot be applied unless there are segments in the problem.

The protocols that were obtained provided an impressive range of performance. Some of the students gave quite clear evidence that their representations of the problems included the part–whole relationships that provide the basis for transfer from segment problems like Fig. 4.3 to angle problems like Figs. 4.1 and 4.2. Other students gave quite clear evidence that these general relationships were not included in their representations. A summary of the six students' performances on Fig. 4.2 is in Table 4.1. Two students, S3 and S5, gave proofs that seemed to result directly from use of the overlap–part–whole schema described previously. These proofs were not correct technically; they involved subtraction of the shared angle $\angle BOC$ before statements about addition of angles were included. This property of the errors is consistent with the hypothesis that the schema was used, because the subtraction procedure is a salient component of the schematic knowledge. For a correct proof, some lower-level operations have to be performed first, and it is plausible to expect that the schema would lead students to think of the top-level operations earlier, and this would lead to the kinds of errors that S3 and S5 committed. Both S3 and S5 gave correct solutions to the segment problem (Fig. 4.3) and remarked about the similarity of that problem and Fig. 4.2.

Two students, S2 and S6, gave correct proofs but did not provide evidence of using the schema. S2 was the student who was working independently and had

worked on proofs about angles. There was no evidence in S2's protocol that the schema was used. S6 worked out a proof that involved an analogy to proofs for segment problems, but it apparently was based on the substitution procedure rather than the structure of part–whole relations. S6 seemed to realize that if expressions involving addition could be found, they could serve as arguments in a procedure that involves substitution, and solved the problem by finding those expressions.

A fifth student, S4, may have used the overlap–part–whole schema for representing the problem, or at least was aware of the general similarity of the structure of Fig. 4.2 and problems with segments. S4 mentioned subtraction as a component of the solution and asked whether the "definition of betweenness" should be used. "Definition of betweenness" was the name given in the course text for addition of lengths of segments, and S4's use of this rather awkward term provided evidence of appreciating the similarity between Fig. 4.2 and corresponding problems that involved segments. S2 did not succeed in solving Fig. 4.2. S2 was asked to solve the problem with segments shown in Fig. 4.3 and had considerable difficulty with it. A reasonable interpretation is that S2's representation of Fig. 4.2 may have included the important part–whole relationships, but that S2's knowledge of the problem-solving operations was too weak to enable a solution to be found for the problem.

Performance on the vertical angles problem, Fig. 4.1, is summarized in Table 4.2. This problem was not given to students S4 and S1.

When the vertical angles problem was presented, S3 gave strong evidence of understanding its relation to Fig. 4.2 and Fig. 4.3, saying "You know something, I'm getting sort of tired of that problem." S3 gave a proof of the vertical angles theorem that was similar to the one that S3 gave for Fig. 4.2, with subtraction incorrectly used before addition of angles was asserted.

S5 did not succeed in finding a proof for the vertical angles theorem. However, there was further evidence that S5 had used the overlap–part–whole relationships in solving Fig. 4.2. When that problem was shown again and S5 was asked

TABLE 4.1
Performance on Fig. 4.2

Students	*Performance*
S3, S5	Conceptually correct proofs with technical errors. Protocol evidence for overlap-part–whole schema—e.g., "I have to subtract." "These are the same."
S2	Correct proof. No protocol evidence for schema.
S6	Correct proof using substitution procedure. Protocol evidence against schema: "I'm trying to find a way that I can substitute."
S4	Failed to find a proof. Protocol evidence for schema: "I would have to subtract . . . would I have to use the definition of betweenness?"
S1	No progress toward proof.

TABLE 4.2
Performance on Fig. 4.1

Student	*Performance*
S3	Proof with technical errors like Fig. 2. Protocol: "It's the same problem again; I'm getting sort of tired of that problem."
S5	Did not find a proof. Saw the way to proceed using subtraction when Fig. 4.2 was shown again.
S2	Correct proof using supplementary angles. Found an analogous proof involving complementary angles for Fig. 4.2
S6	Did not find a proof. Was trapped by perceptual distraction.

whether a similar method could be used for Fig. 4.1, S5 apparently saw that the structure provided a way to proceed, saying, "I could say if I had this, I could subtract the supplementary angle from that one."

S2 gave a correct proof for the vertical angles theorem, using a theorem that two angles that are supplementary to the same third angle are congruent to each other. When S2 was asked whether there was a connection between the vertical angles theorem and Fig. 4.2, S2's response did not provide evidence for use of the overlap–part–whole schema. S2 assimilated Fig. 4.2 to the solution scheme used for the vertical angles theorem, saying that Fig. 4.2 could be solved using complementary angles.

S6 was unable to prove the vertical angles theorem, and gave performance that Wertheimer (1945/1959) noted as evidence for a lack of structural understanding. S6 proceeded with the problem by noting that X + Y = 180° and that W + Z = 180°. There probably are perceptual factors that produce the tendency to think of the problem in this way. S6 was trapped in this view of the problem and never escaped from it. A representation with two distinct pairs of angles is inconsistent with the overlap–part–whole schema, so S6's performance provides further strong evidence that those relationships were not in S6's representation.

In summary, three students gave quite strong evidence that the structure of relationships in the overlap–part–whole schema was in their representation of Fig. 4.2. None of these students gave proofs that were entirely correct; however, S3 and S5 gave proofs that showed a good grasp of the problem. S3 showed good transfer to the vertical angles problem, and S5 recognized the structure of that problem when Fig. 4.2 was shown again. The third of these students, S4, failed to find a proof for Fig. 4.2, apparently because of weak knowledge of problem-solving procedures. A fourth student, S2, solved both Fig. 4.2 and the vertical angles problem successfully, but did not provide any protocol evidence of using the overlap–part–whole schema. It is possible, of course, that the relationships in that structure were recognized and used by S2 in an implicit way. A fifth student, S4, solved Fig. 4.2 successfully and gave relatively clear evidence of not being cognizant of the overlap–part–whole relationships, especially in the

vertical angles problem that S4 was unable to solve. The sixth student, S1, made no progress on any of the problems, apparently lacking a great deal of the knowledge required for successful performance.

Theoretical Analysis

The goal of the theoretical analysis was to identify a set of learning processes and knowledge structures that could simulate acquisition of learning with structural understanding. In developing this model I collaborated with John Anderson; some of the results were presented in Anderson, Greeno, Kline, & Neves (1981) as part of a general discussion of the acquisition of problem-solving skill.

To clarify the specific structures responsible for understanding, two versions of the learning simulation were formulated. One of these is called stimulus–response learning, and the new problem-solving procedures acquired by this system are associated with relatively specific situational cues. In the other system, called meaningful learning, the new problem-solving procedures are acquired as integral parts of schemata that represent problem-solving situations in terms of general part–whole relations.

The models that were formulated simulate learning of procedures from worked-out examples, as do many recent computational models of learning procedures (Neves, 1981; Vere, 1978). The learning that was simulated is based on two tasks that are given early in the geometry course and one task that produces learning that should occur some years earlier.

The first two example problems used for simulation of learning are shown in Fig. 4.4, and the third example is shown in Fig. 4.3. The examples were used to investigate three different aspects of learning in which new material is related to a schema. In the first problem, meaningful learning involves an existing schema and the model simulates learning to apply that schema in a situation where it was not applicable previously. Meaningful learning in the second problem involves elaborating an existing schema by forming new problem-solving procedures that become part of the schema. In the third problem situation, meaningful learning involves building a new schema, which has previously existing schemata as subschemata. In all three cases, rote learning was simulated by a process that acquires problem-solving procedures and associates them with relatively specific representations of the problem situation.

The simulation models were programmed in ACTP (Greeno, 1978), a variant of Anderson's (1976) ACT production system. In this system, there is declarative knowledge represented by a semantic network and procedural knowledge in the form of production rules. In learning from examples, the input for learning is the solution of a problem, and the system learns by adding productions to its procedural knowledge or by adding semantic-network structures to the declarative knowledge that it has available for solving problems. The procedures for learning used in this simulation were not the same as those developed and

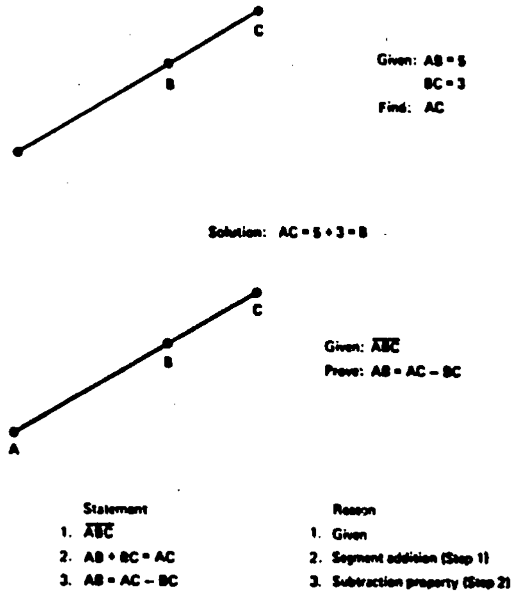


FIG. 4.4. First two example problems for simulation of learning.

investigated by Anderson in his use of ACT as a model of learning (Anderson, 1981).

For learning new productions, a process was developed for keeping a representation of the problem situation in working memory. This representation determines the information that is included in the conditions of new productions when they are added to the system. The representation includes semantic information that is taken from the problem situation (either the diagram or the system's semantic knowledge) that increases the coherence of the problem representation. The process for selecting the added semantic information is similar to that used by Kintsch and VanDijk (1978) for forming coherent representations in their model of text comprehension, although the one developed for ACTP adds more information than Kintsch and VanDijk's does.

Processes also were implemented for schema-based learning. The process for learning to apply an existing schema in a new situation uses the problem representation in working memory to form the condition of the schematizing production, and uses the action performed in the example to determine how to assign components of the situation to slots in the schema. Acquisition of new procedures and synthesis of a new schema required knowledge of general structural features of schemata in the system. This knowledge constitutes a kind of metaschema—a schema that enables the system to acquire new schematic knowledge. For example, procedures attached to schemata include information about consequences that are matched against goals to determine whether use of the procedure is desirable, and information about prerequisites that are tested in the problem

TABLE 4.3
General Components Schemata

Schema
Slots
Schematizing Productions
Contextual Associations
Procedural Attachments
—Prerequisites
—Performances
—Consequences

situation to determine whether the procedure can be applied. The process for acquiring new procedures identifies conditions in the problem situation that are included as prerequisites, and a generalized form of the action in the example is included as a consequence of the procedure. The process for synthesizing a new schema also includes knowledge about the general structure of schemata, in the form of actions that store new semantic knowledge in the form required by the procedures that retrieve and utilize schemata for problem solving.

The schemata that are critical for the system's meaningful learning are included in its declarative knowledge.[2] The structure of schematic knowledge in this model is not unusual; it is a somewhat simplified version of the concept that Bobrow and Winograd (1977) used in the Knowledge Representation Language, (KRL). The main components are listed in Table 4.3. Like all schemata, those used here provide a set of relationships among some objects. The objects fit into the structure in positions called slots. There are procedures for applying the schema in specific situations; these are referred to as schematizing productions. There is some information that identifies features of the situation that are relevant for performing procedures; these are called contextual associations. An important feature of these schemata is that they have procedures associated with them, in the manner of KRL. The organization of these procedure descriptions is patterned after Sacerdoti's (1977) model of planning in problem solving, called Network of Action Hierarchies (NOAH). Information stored about each procedure includes prerequisite conditions, consequences, and specific actions that are carried out in performing the procedure.

Learning New Applicability Conditions. The first example problem that the model learns from is the first problem in Fig. 4.4. I assume that students learn to

[2]I consider the use of declarative structure for representing schemata as a detail of implementation, rather than a substantive psychological hypothesis. In fact, I am inclined to believe that the relationships incorporated in these schemata probably are represented in humans as cognitive procedures, rather than as declarative structures. I believe that a model could be formulated in which schemata would be represented procedurally, and that it would be functionally equivalent to the model that I describe here.

solve problems like this some years before they study geometry. The problem was included in the analysis to clarify the requirement of having knowledge about applying schemata to represent problems.

In the simulation of stimulus–response learning from the first example problem, the action of adding lengths is associated with a representation of the problem situation that specifies some relevant properties such as the collinearity of the segments. The knowledge acquired by rote learning enables the system to solve new problems that are closely similar to the one in which learning occurred, but generalization is very limited. For example, the rote learner does not generalize to a problem in which the total length and one subsegment are given, and the other subsegment is to be calculated.

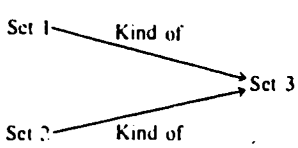
Meaningful learning occurs in the first example problem when there is an active schema that is assumed as prior knowledge: the relationship between parts that are combined to form a whole. The components of this schema are shown in Table 4.4. We have evidence from studies of children's performance on arithmetic story problems (Riley, Greeno, & Heller, 1983) that this schema has been acquired by most 8-year-old children. The schema is used in understanding problems such as the following: "There are five girls and eight children; how many boys are there?" The schematizing production identifies the set of boys and the set of girls as the parts and the set of children as the whole. Then, using the information stored in procedural attachments, the procedure of separation is chosen and the numbers are subtracted to find the answer.

When the system learns meaningfully from the first example problem it learns to apply this schema to problems involving segments. It acquires a new schematizing production, which is sketched in Fig. 4.5. As a result, in future problems like this one the model can interpret the segments as a part–whole structure.

TABLE 4.4
Components of Whole/Parts Schema

Whole/Parts

Slots: Whole, Part 1, Part 2
Context: Set → Number
Schematization:



Procedures:
Combine/Calculate
Separate/Calculate
Adjust Parts

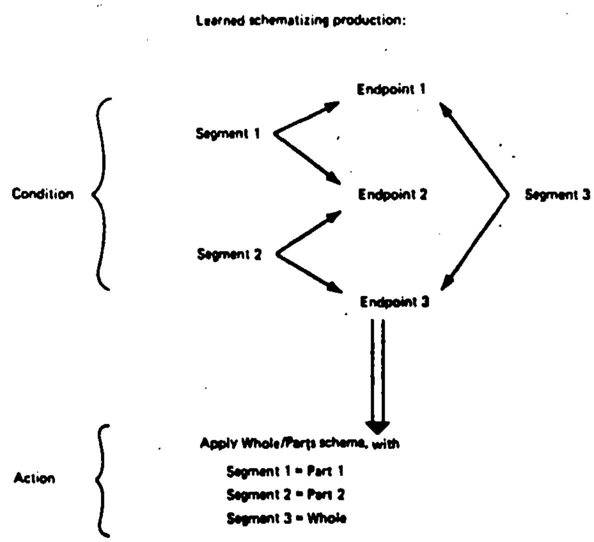


FIG. 4.5. Schematizing production learned from the first example with meaningful learning.

One consequence is that the whole collection of procedures associated with the schema is then available for solving a variety of problems, which is not the case when simple stimulus–response learning occurs. For example. the knowledge acquired in meaningful learning generalizes to problems in which the total length and one subsegment are given and the other subsegment is to be calculated, because when the whole/parts schema is applied, the procedure of separating the whole into parts is made available.

Learning New Procedural Attachments. The second example is the second problem in Fig. 4.4. From this problem the model learns procedures required for writing steps of proof. (Previously, the model's only procedures involved calculations with numbers.) The stimulus–response learning system simply learns actions of writing lines of proof like Steps 2 and 3 in the example when the stimulus conditions are equivalent to those of the example problem, including collinear points and a goal to prove that the length of one segment is the difference between lengths of two other segments.

Meaningful learning from the second example involves an addition of structure to the part–whole schema that is already known by the system. The new structures are procedures that are associated with the schema. They are sketched in Table 4.5. Note that the prerequisites of these procedures include problems that have been schematized as part–whole structures. The actions of the procedures are defined in terms of the slots of the whole/parts schema. This means

TABLE 4.5
Procedures Learned from the Second Example

A. Write-segment-addition:	
Prerequisite:	Whole/Parts schema.
	Statement: Collinear Points
Perform:	Write statement:
	"Part 1 + Part 2 = Whole"
	Write reason: "Segment Addition"
B. Write-subtraction	
Prerequisite:	Whole/Parts schema,
	Statement: Part 1 + Part 2 = Whole
Perform:	Write statement:
	"Part 1 = Whole − Part 2"
	Write reason: "Subtraction Property"

that the actions involve writing steps of proof that refer to objects that occupy the slots of an instantiation of the schema. The main consequence of this learning, in contrast to the stimulus–response learning from this example, is that the procedures that are learned are applicable in any problem situation that is represented using the whole/parts schema. The procedures acquired in stimulus–response learning can only be applied in situations involving segments with collinear endpoints.

Learning a New Schema. The final learning task that was simulated used the problem in Fig. 4.3. Stimulus–response learning from this example acquires the procedures for writing proof steps involving substitution and the subtraction property, based on Steps 5 and 6, and associating them with relatively specific problem conditions, as in the earlier examples.

The meaningful version of learning from this example results in a new schema, sketched in Table 4.6, in which two part–whole schemata are included as

TABLE 4.6
Schema Acquired from Fig. 4.3

Overlap/Whole/Parts
Slots:
Part 1, Part 2, Part 3, Whole 1, Whole 2
Subschemata:
Whole/Parts 1. (Part 1, Part 2, Whole 1)
Whole/Parts 2 (Part 2, Part 3, Whole 2)
Procedures:
Write-Substitution
Write-Subtraction (2 sides)

subschemata. The problem solver represents the segments *RO*, *ON*, and *RN* as a part–whole structure, and also represents the segments *ON*, *NY*, and *OY* as a second part–whole structure. This enables it to apply the procedures for writing steps based on segment addition, corresponding to Steps 3 and 4 of the example. Steps 5 and 6 of the example involve new procedures for the model. They also involve both of the part–whole structures represented in the problem. This leads to the formation of a new schema, with two part–whole structures that share one of their parts. Problem-solving procedures corresponding to Steps 5 and 6 of the example are acquired and associated with the new schema.

Conclusions

The simulation of meaningful learning provides a definite hypothesis about a process that can acquire problem-solving procedures with structural understanding. Although transfer to problems about angles was not implemented, it is clear that the knowledge acquired in meaningful learning from examples about segments is applicable in problems like Figs. 4.1 and 4.2. For a student to make that application, schematizing productions would have to be learned to enable the appropriate schemata to be used to represent relationships among angles, or the student could use more general interpretive processes to comprehend the problems as instances of the overlap–part–whole structure.

It is significant that a plausible simulation of meaningful learning could be formulated using only knowledge structures of relatively standard kinds. Schemata with characteristics that are standard in knowledge representation systems and learning processes that are commonly proposed for learning from examples were combined in a straightforward way to form a simulation of learning with significant structural understanding. This seems quite encouraging for the prospect of developing definite hypotheses about understanding over a substantial range of instructional tasks.

UNDERSTANDING A GENERAL FORMAL PRINCIPLE

The second analysis I present was concerned with a somewhat different aspect of understanding, involving a formal principle. This study, in which I collaborated with Maria Magone, was concerned with geometry students' understanding of the general concept of proof. This constitutes a metaprinciple in relation to the tasks that students are taught to perform in their study of geometry. The skills that students are required to learn involve constructing proofs. The question addressed in this study is whether students understand what proofs are—that is, whether they know the general features that are required for something to be a proof.

The concept we focused on can be called the principle of deductive consequence. In a formal proof, each assertion that is made is deductively required by the premises of the problem or by other assertions that have been made explicitly. This imposes a strong criterion for statements to be acceptable for inclusion in a proof. For each statement that is put into a proof, there must be a sufficient basis in earlier statements that the added statement *must* be true if the earlier statements are accepted. This criterion contrasts with ordinary exposition, in which statements are expected to be reasonable in the light of previous information—for example, new statements should not directly contradict statements made earlier—but there is not a general constraint that each new statement is required by the assertions made previously.

The task that we used to investigate understanding of deductive consequence involved checking proofs. We presented proof problems with solutions worked out, and asked students to check whether each proof was correct. If the student said that there was an error, we asked what was wrong.

Our study had three parts. First, we gave proof-checking problems to some students who were taking a course in geometry. (These were different students than those who gave the protocols described in the first section.) Six problems were given in interviews held in November, and four of the same problems were given in May. Fifteen students participated each time, with ten students participating in both interviews and five additional students who were different in the two interviews. The performance of students on proof-checking tasks was not very good, especially on a problem in which the error to be detected was that an assertion needed to justify a step had been omitted.

The second part of our study was a theoretical investigation. We developed a computational model of correct performance on the proof-checking task, including problems with errors of omitted information. Because the students whom we observed had not performed these tasks successfully, the model we developed was not a simulation of performance that had been observed. However, we attempted to develop a model that was psychologically plausible, in the sense that it used procedural and propositional knowledge that we felt could be acquired by human learners, and would simulate performance of human problem solvers who had acquired that knowledge.

The third part of the study was an instructional experiment to test whether the proof-checking knowledge incorporated in the model could be acquired by human learners. We developed instructional materials for training the procedure of checking proofs. A group of 15 college-student subjects who had studied geometry in high school received this instruction, along with sufficient review of basic geometry to make the instruction possible. A control group was given only the review materials. The experimental group succeeded better on proof checking and other geometry problems than the control group, both in the domain of problems that was used in the instruction and in another domain of problems that was used in a test of transfer.

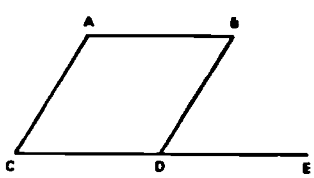
Performance on Proof Checking

One of the proof-checking problems that we used is shown in Fig. 4.6. The proof shown is not valid. For Step 1 to be justified, lines *AC* and *BD* should have been asserted as parallel. Because this was neither given nor proved in prior steps, Step 1 is not justified.

Performance of the geometry students on proof-checking problems is shown in Table 4.7. The problem in Fig. 4.6 was the one that is called "Missing given" in Table 4.7. Note that performance on that problem was poor both in November and in May. Performance on other problems was somewhat better. The valid proof had a diagram in which the statement to be proved appeared false; apparently some students came to disregard the appearance of the diagram between November and May. Other sources of errors included reasons given in the problem that are not theorems of geometry, and reasons whose antecedents did not apply to the objects that the statements referred to. In two problems, the diagrams were consistent with the statements to be proved; in the other two problems, the statements to be proved were incorrect in the diagrams. About half the students found the errors in the problems with consistent diagrams, both in November and in May. Performance in November was somewhat better on the problems where the discrepancy could be seen visually; these problems were not included in the May interviews.

Theoretical Analysis

The goal of the theoretical analysis was to develop a definite hypothesis about knowledge that would enable students to solve proof-checking problems successfully. To do this we formulated a computational model of correct proof-checking performance.



Given: $\overline{AB} \parallel \overline{CE}$ and $\overline{AB} \cong \overline{CD}$
Prove: $\angle ACD \cong \angle ABD$

	Statement	Reason
1	$\angle ACD \cong \angle BDE$	corresponding $\angle$s
2	$\angle BDE \cong \angle ABD$	alternate interior $\angle$s
3	$\angle ACD \cong \angle ABD$	transitive property

FIG. 4.6. A difficult proof-checking problem.

TABLE 4.7
Performance on Proof Checking

Problems	*Number*	*November*	*May*
Valid proof	1	.50	.93
Missing given	1	.06	.19
Other invalid: Diagram consistent	2	.46	.46
Other invalid: Diagram inconsistent	2	.72	—

The procedure that we implemented is sketched in Table 4.8. This procedure is applied to each step in the proof. If all the steps are accepted, and the final step corresponds to the problem goal, then the proof is accepted.

First, the reason that is given in the proof step is checked against a list of theorems that are available for use. (The list includes definitions and postulates as well as theorems; the model does not recognize this technical distinction.) Each theorem is associated with its "deep structure," in which the antecedent and consequent of the conditional proposition are represented with variables that can be matched with objects in the statement and in previous statements and the given information.

The second step of the procedure involves retrieving the antecedent and consequent of the theorem that is given as the reason. Then there is a matching process, in which the consequent of the theorem is matched with the statement and the variables in the consequent are replaced by the objects that are mentioned in the statement. If the consequent cannot be matched to the statement, the step is rejected. This happens if the property or relation asserted in the statement does not match the property or relation of the consequent.

In the fourth step of the procedure, the objects that were found in the statement are substituted for the variables in the antecedent, forming one or more propositions that should have been asserted in previous steps or in the given information. Finally, there is a search for those propositions in the set of previous

TABLE 4.8
Procedure for Verifying a Step

(a) Test whether reason is a theorem.
(b) Retrieve deep structure of reason.
(c) Instantiate consequent in statement.
(d) Form corresponding instantiation in antecedent.
(e) Determine whether antecedent has been asserted.
(f) Determine whether statement matches problem goal.

statements and given information of the problem. If they are found, the proof step is accepted; otherwise, it is rejected.

To clarify the procedure, consider its application to the first step of the problem in Fig. 4.6. First, the reason "Corresponding angles" is checked with the list of available theorems. This is successful; there is a theorem (actually, a postulate) called "Corresponding angles" on the list.

Next, the deep structure of this theorem is retrieved. The antecedent is "If angles X and Y are corresponding angles with lines L and M as sides, and lines L and M are parallel, and line N contains their other sides," and the consequent is, "Then X and Y are congruent."

In the third step, the consequent of the reason is matched with the statement of the proof step. The step says that $\angle ACD$ and $\angle BDE$ are congruent. Therefore, X and Y in the consequent are replaced by these angles.

In the fourth step, the propositions that have to be asserted are formed. This includes a pattern in which $\angle ACD$ and $\angle BDE$ are corresponding angles, with two of their respective sides collinear. There also must be an assertion, either in given information or a previous statement, that the other sides are parallel.

Finally, in a search for these propositions, the system fails to find the assertion about parallel lines that is needed. The system rejects the first step of the proof.

The knowledge required for checking proofs can be compared with knowledge for constructing proofs, especially as a model for solving proof construction problems was formulated earlier (Greeno, 1978). The knowledge needed for proof checking is much simpler in one way than the knowledge that is needed for constructing proofs, but is somewhat more complex in two other ways.

The factor that makes knowledge for checking proofs simpler is that steps of the proof are already shown. This eliminates the need to search for a sequence of problem-solving inferences to achieve the goal of the problem. As a result, the strategic knowledge used for choosing plans and goals that guide the search for inferential operators is not needed for proof checking.

On the other hand, two components of knowledge are required for proof checking that are not needed for constructing proofs. First, the procedures for verifying steps of proof and checking whether the main goal has been satisfied must be added. The other complexity involves the way in which postulates and theorems are represented in the models of proof construction and checking. In the model of proof construction, postulates and theorems are represented implicitly as problem-solving operations in the form of production rules, with the antecedents as conditions that are tested and the consequents as actions that assert properties or relations that are inferred. However, in the model of proof checking there is an explicit representation of postulates and theorems, which are stored in a list so that reasons can be checked to determine whether they are theorems and their antecedents and consequents can be retrieved in the process of determining whether the needed assertions have been made.

Instructional Experiment

We performed a test of our ideas about proof checking by designing some instructional materials based on the computational model of proof checking that we had implemented. This instruction was given to some human learners. The subjects were able to use the procedure that was taught in the instruction, and it resulted in some improvement in their performance in proof checking. We take this as evidence that the model of proof checking has some psychological validity, in the sense that it simulates a cognitive procedure that human problem solvers can learn to perform.

Materials. Instructional materials were designed for presentation to small groups of student subjects. A description was given of a five-step procedure, referred to as the "if–then" procedure because of its emphasis on the antecedent–consequent structure of reasons for proof steps. The steps are: (1) Check if the reason is a valid theorem, postulate, or definition; (2) Divide the reason into its "if" and "then" components; (3) Check the antecedent: has it previously been shown in the given information, the diagram, or in previous steps of the proof? (4) Check the consequent: does it match the relationship of the statement? (5) If the statement is the last one in the proof, does it match the goal of the problem? Note that the steps of the procedure correspond to the steps shown in Table 4.8, except that instantiation of the reason is left implicit. In the instruction, instantiation was shown implicitly in examples, in which the antecedents and consequents of reasons were converted to appropriate propositions about the objects in the problems.

After the procedure was explained, two example problems were worked by the group of student subjects working together, with supervision by the instructor–experimenter. Then booklets were provided that contained proof-checking problems. On each problem, the students first worked individually on the problem, then one student was asked to check the proof aloud, going thorugh the "if–then" procedure for each line of the proof. Any errors or omissions were corrected by the instructor, and questions asked by the students were discussed. Two booklets of proof-checking exercises were prepared for use in two training sessions.

In addition to the materials for instruction in proof checking, there also were materials that reviewed basic concepts of geometry and construction of proofs. Initially, the instructor reviewed basic definitions, postulates, and theorems needed to solve proof problems involving congruence of triangles. Booklets were provided that contained problems of constructing proofs, and students worked individually, then reviewed each problem, as described previously. Two booklets of problems in proof construction were prepared for use in two review sessions. All problems used in the review of proof construction and in training of proof checking were in the domain of congruent triangles.

Materials also were prepared for assessing the results of instruction. A set of problems involving congruence of triangles was prepared as a test, designed to be given individually to the students. The test problems included 10 proof-checking problems and three problems of constructing proofs.

Finally, materials were prepared for testing transfer. These included a brief review of definitions, postulates, and theorems involving angles formed by parallel lines intersected by a transversal. These concepts and propositions had not been included in the previous instruction or testing, and there was no discussion of proof checking during the review of concepts about parallel lines. A transfer test was presented, consisting of 11 proof-checking problems and three problems of constructing proofs, all concerned with angles formed by parallel lines intersected by a transversal.

Subjects and Procedure. Subjects were 30 students at the University of Pittsburgh who had studied geometry in high school but were not majoring either in mathematics or computer science. Subjects were paid for their participation in the experiment.

The study was conducted in 1-hour sessions on consecutive days. Subjects in an experimental condition received the review of triangle congruence and proof construction in two group sessions; then training in checking proofs about congruent triangles was given in two further group sessions; and finally the test problems and the transfer to parallel lines were given in two individual sessions. Subjects in a control condition received the two review sessions and the test and transfer sessions, without the intervening instruction in proof checking. Fifteen subjects (nine women and six men) participated in each of the conditions.

The subjects in the experimental condition participated on 6 consecutive days; subjects in the control condition participated on 4 consecutive days. For each day on which they participated, subjects chose a convenient time from among a number of available sessions. Group sessions usually had three or four students. In the test and transfer sessions, involving individual subjects, students were asked to think aloud as they worked on problems, and their protocols were recorded on audio tape.

Results. Performance of the students in the control and experimental conditions is summarized in Table 4.9. The instruction apparently had a positive effect, enabling students in the experimental condition to detect a higher proportion of errors than the control students. The difference in performance on invalid proofs was significant ($t(28)$ 2.35, $p < .05$), and there clearly was not an interaction between training conditions and sessions. The data in Table 4.9 are based on strict scoring of error detection, in which errors had to be identified and the reasons for the errors given correctly. Data also were tabulated using a more lenient criterion, where errors had to be identified correctly but the subject's

TABLE 4.9
Proportion Correct

Problem Type	*Session*	*Number of Problems*	*Control*	*Experimental*
Invalid	Test	7	.42	.62
	Transfer	8	.50	.70
Valid	Test	3	.87	.89
	Transfer	3	.87	.89
Proof construction	Test	3	.20	.53
	Transfer	3	.33	.49

explanation of the error was incorrect in some way. With the lenient criterion, there was a significant main effect of training ($t(28)$ 2.89, $p < .01$) and a nonsignificant interaction with sessions ($t(28) = 0.59$, $p > .50$), as with the strict criterion.

An examination of the individual problems failed to reveal any systematic differences in the kinds of problem for which the instruction in proof checking was especially effective. For example, some errors in proof involve incorrect reasons; others involve antecedents that have not been established. There was a somewhat larger effect of instruction on problems with incorrect reasons in the test session, but the effect on problems with antecedents not established was greater in the transfer session. One specific problem for which instruction had an especially large effect is the problem shown in Fig. 4.6, which was included in the transfer session. One of the 15 control subjects detected the error in this problem correctly, whereas eight of the experimental subjects did.

The thinking-aloud protocols were examined to determine whether the subjects used the proof-checking procedure in an explicit way. Three of the subjects in the experimental condition used the procedure on nearly all the steps in every problem. Most of the students in the experimental condition applied the procedure when they appeared to be uncertain about a step. Only two of the subjects in the control condition used a form of the proof-checking procedure that included explicit checking of the antecedents and consequents of reasons.

The experimenters formed two general impressions of differences between subjects in the two conditions. First, experimental subjects appeared to read the statements of the proof more carefully than did the control subjects, even when the experimental subjects were not using the proof-checking procedure explicitly. Second, the control participants typically checked steps by comparing the statements with the problem diagrams. Only one student in the experimental condition appeared to use the strategy of comparing statements with diagrams. The rest attended primarily to the reason and related it to information from previous steps in the proof.

There was a substantial difference between the two conditions in performance on proof construction problems in both the test and transfer sessions. The difference was significant ($t(28)$ 2.94, $p < .01$) and the interaction with sessions was not significant ($t(28) = 1.02$, $p > .50$). A substantial part of the difference was due to one problem in the test session that used a pattern of overlapping triangles on which experimental subjects had some specific practice that was not given to control subjects. Even without that problem, however, there was an advantage for the experimental subjects: .43 to .30 on proof construction problems. It seems likely that the training in proof checking gave subjects some skills that facilitated their performance in proof construction problems as well.

Discussion. The subjects who were given training in proof checking did not become experts on that task, but they clearly acquired some skill that subjects without that training lacked. The result provides general support for the computational model of proof checking, showing that it is learnable at least to an extent. The fact that students still failed to detect some errors may be attributable simply to the relatively small amount of training, combined with the absence of any constraints on the subjects to use the procedure they had learned in the test and transfer problems. As a methodological point, the experiment has the interesting feature that human problem solvers were trained to perform in agreement with a model that existed earlier; thus, we succeeded in getting human performance to simulate a computer program, rather than the other way.

An important conceptual question is whether the skill represented in the computational model and acquired to some extent by the student subjects constitutes significant understanding of the concept of proof. It seems to us that it does. Knowledge of the procedure enables an individual to analyze relations between each step of a proof and the information in previous statements and the premises of the problem. The relation that is examined is deductive consequence, the defining characteristic of formal proof. It seems reasonable to characterize a procedure for testing the features of deductive consequence as a form of significant implicit understanding of the concept of formal proof.

The view that ability to check proofs reflects understanding of the principle of deductive consequence is strengthened by contrasting correct performance on proof checking with performance by students who have not received instruction in the task. Typical untutored performance strongly resembles comprehension of narrative or expository text, where each new statement is accepted if it is consistent with previous information and can be added to it in a coherent structure. In a proof, a stronger criterion of acceptance should be applied to each new statement: It must not only be consistent with prior information, it must follow from it deductively. The procedure for checking applies this stronger criterion, and knowledge of the procedure therefore provides a form of knowledge that relates directly to characteristics that distinguish formal proof from ordinary texts.

CONCLUSIONS

In the introduction to this chapter I mentioned four general points that are illustrated by aspects of the analyses that I have discussed. I now return to those points as a framework for presenting some conclusions.

First, the analyses illustrate the applicability of methods of analysis developed in cognitive science to school tasks, and show how those methods can lead to formulation of cognitive objectives of instruction. The analyses summarized by Resnick (this volume) involving reading, physics, and elementary mathematics, and the investigations reported by Scardamalia and Bereiter (this volume) involving writing, should leave little doubt as to the feasibility of research into the cognitive requirements of significant instructional tasks using currently available concepts and methods of cognitive science. Of course, this should not be interpreted as suggesting that more powerful and valid concepts and methods will not be developed in the future, but only that significant and useful insights can be obtained with the tools that we have at present.

These analyses also provide characterizations of knowledge that can be adopted as definite objectives of instruction. The analysis of structural understanding given here identifies a specific cognitive structure of general relationships as the knowledge that constitutes understanding of the structure of a class of geometry problems. The analysis of understanding of the principle of deductive consequence identifies a cognitive procedure that incorporates defining features of valid deductive inference. Both these characterizations could provide objectives for instruction that are specific enough to be incorporated into instructional materials, if it is thought that their acquisition would be valuable.

The second general point involves analysis of acquisition and instructional intervention. In the example of structural understanding, a theoretical analysis of acquisition was developed. This analysis was ad hoc in many ways, and much work remains for the development of a general model of schema-based learning. Even so, some suggestions for instruction can be taken from the analysis. A major condition for meaningful learning to occur in the model is the activation of an appropriate schema in the learner's knowledge structure. This suggests that at a minimum, instruction with the goal of structural understanding should include an effort to activate general schematic knowledge that can provide an appropriate structure for the material being learned. A straightforward method would involve simple discussion of the structure involved; in the case of the part–whole schema, pointing out to students that segments or angles in the problems have other segments or angles as their parts, and discussing the way in which these part–whole structures are involved in the inferences that are made in solving the problems. Another instructional method involves structural mapping, where an analogy is used involving the problem that is the target of instruction and a problem from another domain that has the same structure in a salient form. The efficacy of analogical mapping for geometry problems has not been studied

systematically;[3] however, its general usefulness is widely recognized, and it has been studied systematically in other domains such as electricity and electronics (Gentner, 1980; Riley, Bee, & Mokwa, 1981) and in elementary mathematics (Resnick, in press, this volume).

In the example of understanding the principle of deductive consequence, instructional materials were developed for teaching the procedure of proof checking that incorporates defining features of valid deductive inference. Instruction using the materials was modestly successful, and because we gave only 2 hours of instruction, it seems quite likely that sustained use of the method in a geometry course would have substantial benefit for students. This example, involving an instructional objective in procedural form, is particularly adaptable for instructional purposes, as a procedure can both be taught and assessed in a straightforward way.

The third general point involves theoretical significance of the analyses. Development of definite models as hypotheses about cognitive structures that constitute understanding should provide some clarification of understanding in relation to general concepts of psychological theory. The major psychological concept involved in the example of structural understanding is that of a schema. This concept has been central in the theory of language understanding that has been developed recently (e.g., Norman & Rumelhart, 1975), and it would be reasonable to expect that it would also be useful in the analysis of understanding in nonlinguistic domains. A satisfying conceptual continuity is provided by the finding that a schema organized in essentially the same way as those hypothesized in analyses of language understanding and knowledge representation (Bobrow & Winograd, 1977) can provide a plausible account of structural understanding of a class of mathematical problems.

The theoretical contribution of the analysis of understanding the principle of deductive consequence involves a characterization of implicit understanding. We attribute implicit understanding of a principle to an individual when the principle plays a significant functional role in the individual's performance. One way in which a principle can be functional is in a procedure for evaluating examples to determine whether they satisfy the principle. The procedure for proof checking that was formulated and that students acquired in instruction evaluates solutions

[3]I can report an anecdote in which the method of analogical mapping provided a successful instructional experience for the vertical angles problem. An eighth-grade student was shown the proof of the theorem, but was unable to recall it about two weeks later. Two analogous situations were then discussed. One of these involved distances on a map, with two equal total distances and a shared component, and the conclusion of equal partial distances with the shared component removed. The other situation involved a person standing on a scale holding one of two filled suitcases; the combined weights of the person with either of the suitcases was equal, and the conclusion was that the suitcases had to be equal. With these items of background, the student generated the proof of equal vertical angles and remembered it on two later occasions, one 2 days later and the other 7 months later.

of proof problems according to the principle of deductive consequence, and thus illustrates this form of implicit understanding.

Using performance in an evaluation task such as proof checking as a criterion for knowledge of a general concept has two important precedents. First, in experimental studies of concept identification, subjects' ability to distinguish correctly between positive and negative examples is taken as the criterion of their acquisition of the concept. In a more complex domain, knowledge of the grammar of a language is tested by the ability of a human subject or computational system to discriminate correctly between strings that are sentences of the language and strings that are not sentences according to the grammar.

The final general point mentioned in the introduction was that "understanding" refers to numerous distinct forms of knowledge. The examples discussed in this chapter illustrate two major categories of understanding: intrinsic and theoretical understanding. Intrinsic understanding of a problem involves cognizance of relationships among elements of the problem or steps in its solution. Theoretical understanding involves cognizance of relationships between the problem and general principles that constrain or justify aspects of the solution. (A more extended discussion of these forms of understanding has been given by Greeno & Riley, 1981.) Structural understanding, in the characterization given here, is a form of intrinsic understanding, involving cognizance of a set of relationships among problem components. Understanding the principle of deductive consequence is an example of theoretical understanding, involving cognizance of a significant constraint on valid solutions of proof problems.

The difference between these two forms of understanding emphasizes the importance of identifying the cognitive structures and processes that are intended when we ask whether someone understands a problem or a procedure. The cognitive structures identified in each of the analyses constitute significant understanding of proof problems. We would say that a student with cognizance of the part–whole relations in the vertical angles problem has more understanding of that problem than a student without that cognizance. Similarly, a student with cognizance of the defining features of valid deductive inference has more understanding of any proof problem than a student without that cognizance. Furthermore, there are other significant forms of understanding in the domain of proof problems that could be taken into account in a theoretical analysis or in design of instruction.

Although an undifferentiated concept of understanding is inadequate for both theoretical and instructional purposes, a more differentiated and specific characterization of understanding is both important and feasible. Concepts and methods of cognitive psychology have now been developed that make it possible to formulate objectives for instruction that are specific enough to be used as the basis of instructional design and assessment, and that also correspond to significant forms of understanding.

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REFERENCES

Anderson, J. R. *Language, memory, and thought*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1976.

Anderson, J. R. *Acquisition of cognitive skill*. (Technical Report 81-1). Pittsburgh: Carnegie-Mellon University Department of Psychology, 1981.

Anderson, J. R., Greeno, J. G., Kline, P. J., & Neves, D. M. Acquisition of problem-solving skill. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Bobrow, D. G., & Winograd, T. An overview of KRL, a knowledge representation language. *Cognitive Science*, 1977, *1*, 3-46.

Gelman, R., & Gallistel, C. R. *The child's understanding of number*. Cambridge, Mass.: Harvard University Press, 1978.

Gentner, D. *The structure of analogical models in science*. (Report No. 4451). Cambridge, Mass.: Bolt, Beranek, & Newman, 1980.

Greeno, J. G. Indefinite goals in well-structured problems. *Psychological Review*, 1976, *83*, 479-91.

Greeno, J. G. A study of problem solving. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 1). Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Greeno, J. G., Magone, M. E., & Chaiklin, S. Theory of constructions and set in problem solving. *Memory & Cognition*, 1979, 7, 445–461.

Greeno, J. G., & Riley, M. S. *Processes and development of understanding*. (Report 1981-8). Pittsburgh: University of Pittsburgh Learning Research and Development Center, 1981.

Judd, C. M. The relation of special training to general intelligence. *Educational Review*, 1908, *36*, 28-42.

Katona, G. *Organizing and memorizing*. New York: Columbia University Press, 1940.

Kintsch, W., & VanDijk, T. A. Toward a model of text comprehension and production. *Psychological Review*, 1978, *85*, 363-394.

Neves, D. M. *Learning procedures from examples*. Unpublished doctoral dissertation, Carnegie-Mellon University, 1981.

Norman, D. A., & Rumelhart, D. E. *Explorations in cognition*. San Francisco: W. H. Freeman, 1975.

Piaget, J. *The child's conception of number*. New York: W. W. Norton, 1965. (Originally published 1941.)

Resnick, L. B. A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *Development of mathematical thinking*. New York: Academic Press, in press.

Riley, M. S., Bee, N. V., & Mokwa, J. J. *Representations in early learning: The acquisition of problem-solving strategies in basic electricity/electronics*. Pittsburgh: University of Pittsburgh Learning Research and Development Center, 1981.

Riley, M. S., Greeno, J. G., & Heller, J. I. *Development of word problem-solving ability*. In H. P. Ginsburg (Ed.), *Development of mathematical thinking*. New York: Academic Press, 1983.

Sacerdoti, E. D. *A structure for plans and behavior*. New York: Elsevier-North Holland, 1977.

Vere, S. A. Inductive learning of relational productions. In D. A. Waterman & F. Hayes-Roth (Eds.), *Pattern-directed inference systems*. New York: Academic Press, 1978.

Wertheimer, M. *Productive thinking* (Enlarged ed.). New York: Harper & Row, 1959. (Originally published 1945.)